MATHEMATICS

1. Let A = three black balls are drawn E_i = Bag contains i white and 10 – i black balls

$$P(E_1/A) = \frac{P(A\,|\,E_1)P(E_1)}{\sum_{i=0}^{10} P(A\,|\,E_i)P(E_i)} = \frac{\frac{1}{11}\times\frac{{}^9C_3}{{}^{10}C_3}}{\frac{1}{11}\left(\frac{{}^{10}C_3 + {}^9C_3 + \ldots\ldots + {}^3C_3}{{}^{10}C_3}\right)} = \frac{{}^9C_3}{{}^{11}C_4} = \frac{14}{55}$$

2. Total ways = 3^{10} Favorable ways = $3^{10} - {}^3C_1 \times 1 - {}^3C_2\,(2^{10} - 2)$

3. Vowels I, I, O are at place (1, 2, 3), (2, 3, 4), (3, 4, 5), (4, 5, 6), (1, 3, 5), or (2, 4, 6)

$\Rightarrow \quad 6 \times \frac{3!}{2!} \times 3! = 108$

4. Sum = $1({}^{21}C_{10}) + 2({}^{21}C_{10}) + \ldots\ldots + 22({}^{21}C_{10})$

5. $$P(A|E) = \frac{P(A\cap E)}{P(E)} = \frac{\sum_{i=0}^{4} P(E_i)\,P(A\,|\,E_i)}{\sum_{i=0}^{4} P(E_i)} = \frac{\frac{{}^{48}C_{26}}{{}^{52}C_{26}}\times\frac{{}^4C_3}{{}^{26}C_3} + \frac{{}^4C_1\,{}^{48}C_{25}}{{}^{52}C_{26}}\times\frac{{}^3C_3}{{}^{26}C_3} + 0 + 0 + 0}{\frac{{}^4C_0\,{}^{48}C_{26} + {}^4C_1\,{}^{48}C_{25} + {}^4C_2\,{}^{48}C_{24} + {}^4C_3\,{}^{48}C_{23} + {}^4C_4\,{}^{48}C_{22}}{{}^{52}C_{26}}}$$

$$= \frac{4\left({}^{48}C_{26} + {}^{48}C_{25}\right)}{{}^{26}C_3\,({}^{52}C_{26})} = \frac{4(49!)3!\,23!\,26!\,26!}{26!\,23!\,26!\,52!} = \frac{4(3!)}{52\times51\times50} = \frac{1}{13.17.25}$$

6. × × × × × T H H

First five are (no consecutive heads) 5T or 4T, 1H or 3T, 2H or 2T, 3H

i.e. T T T T T T H H

or H T T T T T H H or H T H T T T H H or H T H T H T H H

$\Rightarrow$ Required probability = $\frac{1 + {}^5C_1 + {}^4C_2 + {}^3C_3}{2^8} = \frac{13}{256}$

7. $P(A\,|\,B_1) = 0.6$, $P(A\,|\,B_2) = 0.8$

$P(B_1) = 0.3$, $P(B_2) = 0.7$

$$P(A) = \sum_{i=1}^{2} P(B_i)\;P(A\,|\,B_i) = \frac{3}{10}\times\frac{6}{10} + \frac{7}{10}\times\frac{8}{10} = 0.74$$

8. P(cards are higher or lower in rank) = $\frac{{}^{13}C_2\,{}^4C_1\,{}^4C_1}{{}^{52}C_2} = \frac{16}{17} \quad \Rightarrow \quad P(\text{same}) = \frac{1}{17}$

As P(H) + P(L) + P(same) = 1 $\Rightarrow$ P(H) = P(L) = $\frac{8}{17}$

9. Total ways = $2({}^8C_3) + 4({}^7C_3 + {}^6C_3 + {}^5C_3 + {}^4C_3 + {}^3C_3) = 392$

10. Total words formed = $\frac{8!}{4!\,2!\,2!} = 420$

Let ABBC = ×

Number of ways in which × ABBC can be arranged = $\frac{5!}{2!}$ = 60 but this includes ×ABBC and ABBC×.

But the word ABBCABBC is counted twice in 60 hence it should be 59 so required number of ways = 420 – 59 = 361

11. No. of functions = ${}^{8}C_3 \times ({3}^{5} - {}^{3}C_1 2^5 + {}^{3}C_2\, 1^5) = 8400$

12. He has 3, 2, 2, 2, 1 ways respectively at the end of 1 minute, 2 min, 3min, 4 min and 5 min
so 3 × 2 × 2 × 2 × 1 = 24 ways

13. $n(2 \cup 3) - n(3 \cap 4) - n(2 \cap 4) + n(2 \cap 3 \cap 4) = 67 - 8 - 25 + 8 = 42$

14. No. of ways = 2(No. of ways to express 20! as product of two co-prime factors)= $2(2^{n-1}) = 2^n = 2^8 = 256$

15. ${}^{5}C_2 \times 4$

16. $p_n = \frac{p_{n-1}}{2} + \frac{p_{n-2}}{4}$, $n \ge 4$

| | T | H | $= p_{n-2} \times \frac{1}{4}$ | | | T | $= p_{n-1} \times \frac{1}{2}$

As $p_2 = \frac{3}{4}$ and $p_3 = \frac{5}{8}$ $\therefore$ By above formula, $p_4 = \frac{8}{16}$

similarly $p_5 = \frac{13}{32}$, $p_6 = \frac{21}{64}$, $p_7 = \frac{34}{128}$, $p_8 = \frac{55}{256}$, $p_9 = \frac{89}{512}$, $p_{10} = \frac{144}{1024}$

17. Digits to be used are ≥ 6
999996 $\Rightarrow$ 6 ; 999987 $\Rightarrow$ 30 ; 999888 $\Rightarrow$ 20 $\therefore$ total = 56

18. Required number = $\frac{8!}{5! \times 3!} + \frac{8! \times 2!}{4! \times 4! \times 2!} = 126$

19. Required probability = P(A & C throw same number) + P(A & C throw different number)

$= \frac{6 \times 5 \times 1 \times 5}{6^4} + \frac{6 \times 4 \times 5 \times 4}{6^4} = \frac{150 + 480}{1296} = \frac{630}{1296}$

20. Required number of ways = ${}^{16}C_{10} - {}^{9}C_9\, {}^{7}C_1 = {}^{16}C_6 - 7$

21. $|aw + b|^2 = 1 \Rightarrow a^2 - ab + b^2 = 1 \Rightarrow (a - b)^2 + ab = 1$

($a^2 - ab + b^2 = 1 \Rightarrow$ ab cannot be negative integer)

When $(a - b)^2 = 0$ then ab = 1 $\Rightarrow$ (a, b) = (1, 1), (–1, –1)

When $(a - b)^2 = 1$ then ab = 0 $\Rightarrow$ (a, b) = (0, 1), (1, 0), (0, –1), (–1, 0)

22. $(2k_1 + 1) + (2k_2 + 1) + (2k_3 + 1) + (2k_4 + 1) = 32$

$\Rightarrow k_1 + k_2 + k_3 + k_4 = 14 \Rightarrow {}^{14+4-1}C_{4-1} = {}^{17}C_3 = 680$

23. $x^3 + ax^2 + bx + c = (x^2 + 2)(x + a) + (b - 2)x + (c - 2a) \Rightarrow b = 2 \;\&\; c = 2a$

24. P(score of 5) = P(5) + P(14) + P(113) + P(1112) $= \frac{1}{6} + \frac{1}{6^2} + \frac{1}{6^3} + \frac{1}{6^4}$

P(score of 8) = P(116) + P(1115) + P(11114) + P(111113) + P(1111112)
$= \frac{1}{6^3} + \frac{1}{6^4} + \frac{1}{6^5} + \frac{1}{6^6} + \frac{1}{6^7}$

25. $P\left(\frac{B}{A \cup B^c}\right) = \frac{0.2}{0.8} = \frac{1}{4}$

$P(A/B) = \frac{0.2}{0.4} = \frac{1}{2} \Rightarrow P(A/B^c) = \frac{0.5}{0.6} = 5/6$

26. P(5 persons has same selection) = ${}^6C_5 \times {}^4C_1 \times \left(\frac{1}{4}\right)^5 \left(\frac{3}{4}\right)^1$

P(6 persons has same selection) = ${}^4C_1 \times \left(\frac{1}{4}\right)^6 = \frac{1}{4^5}$ Also $P\left(\frac{A_5}{A_6}\right) = 0$

27. $P(A \cup B) \le 1 \Rightarrow P(A) + P(B) - P(A \cap B) \le 1 \Rightarrow P(A \cap B) \ge P(A) + P(B) - 1$

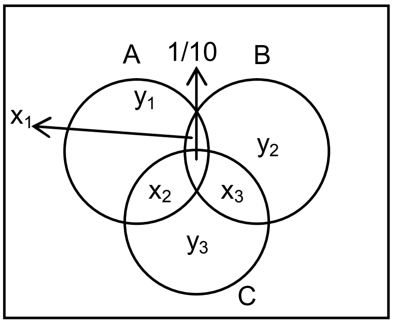
28.

x	P(x)
2	$\frac{3}{36}$
3	$\frac{8}{36}$
4	$\frac{14}{36}$
5	$\frac{8}{36}$
6	$\frac{3}{36}$

29. $P(A \cap B \cap C) = \frac{1}{2} - \frac{2}{5} = \frac{1}{10}$

$x_1 + x_2 + x_3 = \frac{2}{5}$

$y_1 + y_2 + y_3 = \frac{3}{4} - \frac{2}{5} - \frac{1}{10} = \frac{1}{4}$



$\therefore P(A) + P(B) + P(C) = P(A \cup B \cup C) + P(AB) + P(BC) + P(CA) - P(ABC)$

$= \frac{3}{4} + \frac{7}{10} - \frac{1}{10} = \frac{27}{20}$

30.

1R	1R	1R	1R	1R	1R	→	red marbles in the box.
B_1	B_2	B_3........B_k		B_{2009}	B_{2010}	→	boxes
1W	2W	3W	kW	200W	2010W	→	white marbles in the box.

Now P(n) = probability that child stops after drawing exactly n marbles.

i.e. at the n^{th} position red marble must be drawn.

$\therefore \quad P(n) = \left(\frac{1}{2}\right)\left(\frac{2}{3}\right)\left(\frac{3}{4}\right)......\left(\frac{n-2}{n-1}\right)\left(\frac{n-1}{n}\right)\underbrace{\left(\frac{1}{n+1}\right)}_{\text{red}} = \frac{1}{n(n+1)}$

$\therefore \quad \frac{1}{n(n+1)} < \frac{1}{2010} \Rightarrow \frac{2}{n(n+1)} < \frac{1}{1005} \Rightarrow \frac{n(n+1)}{2} > 1005$

$\Rightarrow \quad n \ge 45 \Rightarrow$ B, C, D

31. $P(A) = \dfrac{\dfrac{10!}{2!\,2!}}{\dfrac{11!}{2!\,2!\,2!}} = \dfrac{2}{11} = P(B) = P(C)$

$P(A \cap B) = \dfrac{\dfrac{9!}{2!}}{\dfrac{11!}{2!\,2!\,2!}} = \dfrac{2}{55} = P(A \cap C) = P(B \cap C)$

$P(A \cap B \cap C) = \dfrac{8!}{\dfrac{11!}{2!\,2!\,2!}} = \dfrac{4}{495}$ $\Rightarrow$ $P\,((A \cap \bar{B}) \mid \bar{C}) = \dfrac{\dfrac{2}{11} - \dfrac{2}{55} - \dfrac{2}{55} + \dfrac{4}{495}}{1 - \dfrac{2}{11}} = \dfrac{58}{405}$

32. $A_n - A_{n-1} = 25!\,({}^{49}C_{25})$

$\Rightarrow$ ${}^{n}C_{25}(25!) - {}^{n-1}C_{25}(25!) = 25!\,({}^{49}C_{25})$ $\Rightarrow$ ${}^{n}C_{25} - {}^{n-1}C_{25} = {}^{49}C_{25}$

$\Rightarrow$ ${}^{n-1}C_{24} = {}^{49}C_{24}$ $\Rightarrow$ $n - 1 = 49$ $\Rightarrow$ $n = 50$

33. ${}^{n}C_2 - n = n + k,\ k \geq 10$

$\Rightarrow$ $\dfrac{n(n-1)}{2} = 2n + k$ $\Rightarrow$ $n^2 - n = 4n + 2k$ $\Rightarrow$ $n^2 - 5n = 2k$

$\Rightarrow$ $\left(n - \dfrac{5}{2}\right)^2 = 2k + \dfrac{25}{4} \geq \dfrac{105}{4}$ $\Rightarrow$ $n - \dfrac{5}{2} \geq \dfrac{\sqrt{105}}{2}$ $\Rightarrow$ $n \geq \dfrac{5 + \sqrt{105}}{2}$ $\Rightarrow$ $n \geq 7$

34. Number of divisors = $(n - 1 + 1)(1 + 1) = 2n$ $\Rightarrow k = 2n - 1$

35. Divisors of N are $1, 2, 2^2, \ldots\ldots, 2^{n-1}, 2^n - 1, 2(2^n - 1), 2^2(2^n - 1), 2^{n-1}(2^n - 1)$

$\therefore$ $1 + \dfrac{1}{\alpha_1} + \dfrac{1}{\alpha_2} + \ldots\ldots + \dfrac{1}{\alpha_k} = \left(1 + \dfrac{1}{2} + \dfrac{1}{2^2} + \ldots\ldots + \dfrac{1}{2^{n-1}}\right) + \dfrac{1}{2^n - 1}\left(1 + \dfrac{1}{2} + \dfrac{1}{2^2} + \ldots\ldots + \dfrac{1}{2^{n-1}}\right)$

$= 2\left[1 - \left(\dfrac{1}{2}\right)^n\right]\left(1 + \dfrac{1}{2^n - 1}\right) = 2\left(\dfrac{2^n - 1}{2^n}\right)\left(\dfrac{2^n}{2^n - 1}\right) = 2$

36. Number of ways = $2^{p-1} = 2^{2-1} = 2^1 = 2$

37. ${}^{n}C_6 = 6\ {}^{n}C_3$ $\Rightarrow$ $(n - 3)(n - 4)(n - 5) = 10 \times 9 \times 8$ $\Rightarrow$ $n = 13$

38. Let $P(A) = \sin^2\theta$ $\Rightarrow$ Given expression = $3\sin\theta + 4\cos\theta$ whose maximum value is 5.
where $0 \leq \theta \leq 90°$

39. $f(x) - f(-x) = 6x$ $\Rightarrow$ $f(4) - f(-4) = 24$

$\Rightarrow$ $N = 2310 = 2.3.5.7.11$

Hence number of divisors = $2^{n-1} = 2^{5-1} = 16$

40.

	Category	Selection	Arrangement
1.	All 4 alike	1	$=1$
2.	3 alike + 1 different	$2 \times {}^{3}C_1 = 6$	$6 \times \dfrac{4!}{3!} = 24$
3.	2 alike + 2 different	${}^{3}C_1 \times {}^{3}C_2 = 9$	$9 \times \dfrac{4!}{2!} = 108$
4.	2 alike + 2 other alike	${}^{3}C_2 = 3$	$3 \times \dfrac{4!}{2!2!} = 18$
5.	All 4 different	1	$4! = 24$
			Total = 175